3.9: Antiderivatives

$g(x) = \sin x$ ----------differentiate---------→ $g'(x) =$ ______________

<---------- ------------

$g(x) =$ ________ -------------------------------------→ $g'(x) = 3x^2$

<-------------------------------------

$g(x)$ is called an ________________________ of $g'(x)$

Notation: Generally asked in this way: Given that $f(x) = \cos x$, find *an* antiderivative. That is, find $F(x)$ such that $F'(x) = f(x)$. Using this notation,

$F(x)$ is called *an* antiderivative of $f(x)$.

So $F(x) = x^3$ is *an* antiderivative of $f(x) = 3x^2$. Are there others?

From theorems I asked you to read in section 3.2, which are proved via MVT:

> **5 Theorem** If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

> **7 Corollary** If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

So $F(x)=x^3$ is *an* antiderivative of $f(x)=3x^2$, $F(x)=x^3+C$ is *the most general* antiderivative of $f(x)=3x^2$ (family of curves)

Exampe: Find the general antiderivative for each of the following:

$f(x)=\cos x$ ____________________

$f(x)=\sec x\tan x$ ____________________

$f(x)=x^4$ ____________________

$f(x)=3$ ____________________

$f(x)=x^n$ ____________________

$f(x)=\sqrt{x}$ ____________________

$f(x)=\dfrac{1}{x}$ ____________________

$f(x)=\tan x$ ____________________

$f(x)=x^5+x^3$ ____________________ (Properties)

$f(x)=3x^6$ ____________________

$f(x)=\dfrac{7x^3-\sqrt{x}}{x^2}$ ____________________

Example: Finding a specific antiderivative: Find $f(x)$ given that $f'(x) = 3 - x^{2/3}$ and $f(1) = 0$

Application of Antiderivatives: Rectilinear Motion

In Chapter 2 we learned that if an object is moving in a straight line having position given by $s(t)$:

$$s(t)$$

$$v(t) = s'(t)$$

$$a(t) = v'(t) = s''(t)$$

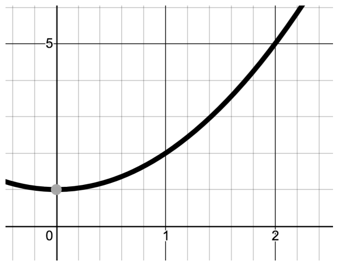
Chapter 4 : Integrals

4.1. Motivation for Integration – Two Classic Problems

1)__________________________________ 2)__________________________

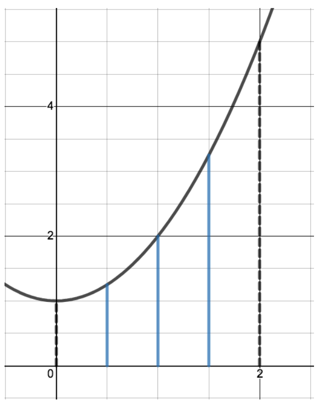
<u>The Area Problem:</u> Given f(x) ≥0, continuous on I=[a, b]. Find the area below f(x) and above the x axis, over I.

<u>Example</u>: Find the area below $f(x) = x^2 + 1$, over [0, 2]

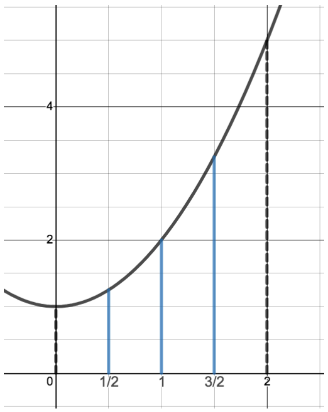


Suppose we cut the region into 4 vertical strips of equal width. What would the width be? ____________________ How do we find the x value for the location of each of these strips? These strips could be approximated by rectangles. How do we decide where to draw the tops of the approximating rectangle?

Left



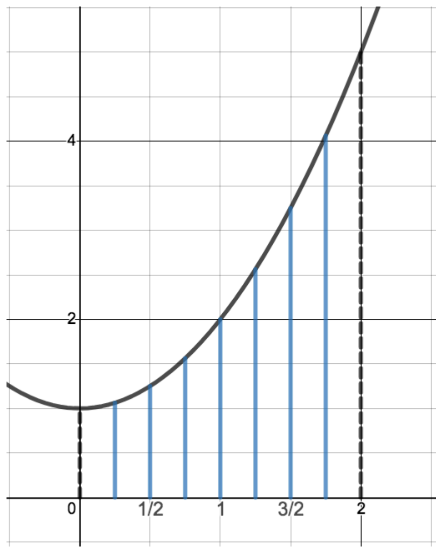
Right



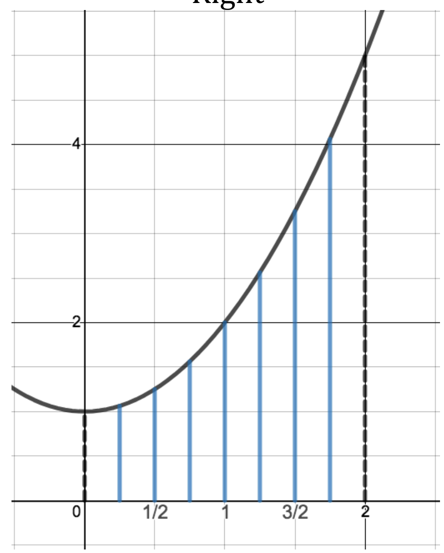
How could we get a better approximation?

Consider 8 vertical strips of equal width. What is the width? X values?

Left



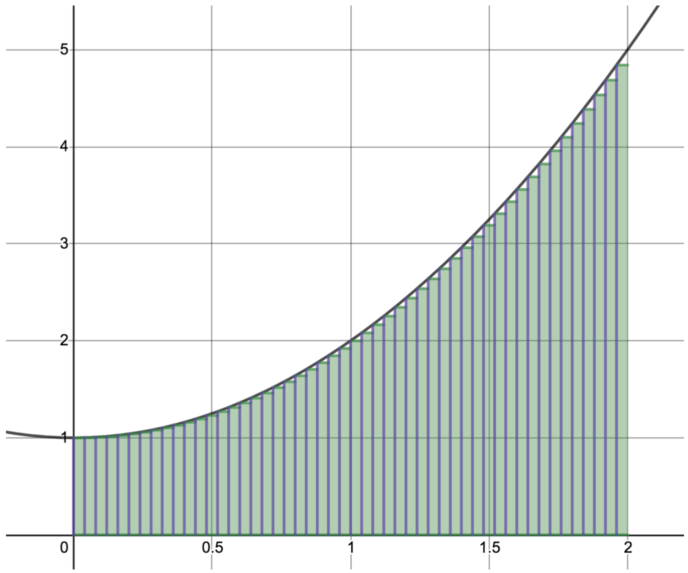
Right



$A \approx L_8 = 4.1875$

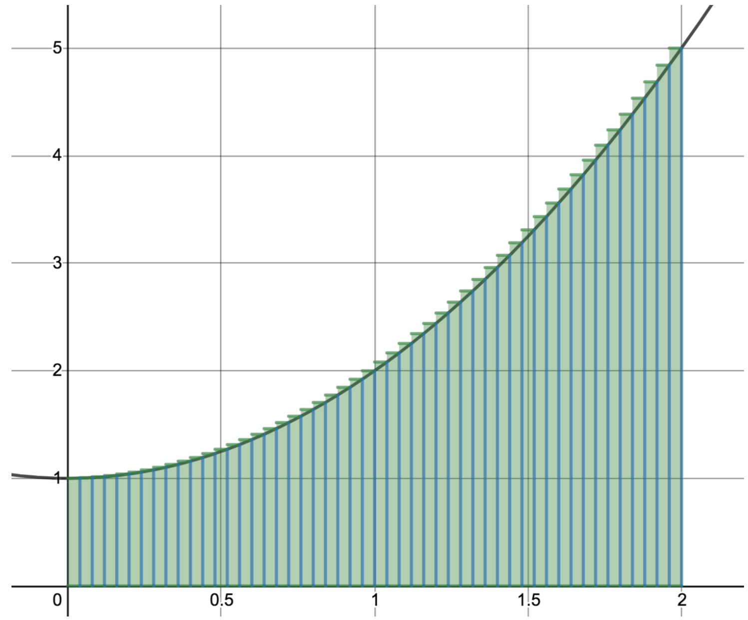
$A \approx R_8 = 5.1875$

https://www.desmos.com/calculator/yfs11mco2v



$A \approx L_{50} = 4.5872$

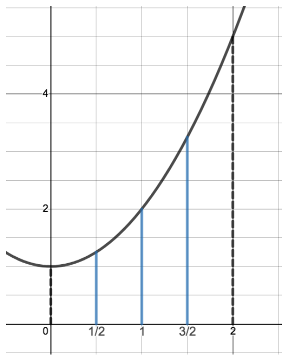
Number of Terms	Riemann Sum
4	3.75
8	4.1875
16	4.421875
32	4.542969
64	4.604492
128	4.635498
256	4.651062
512	4.658859
1024	4.662762
2048	4.664714



$A \approx R_{50} = 4.7472$

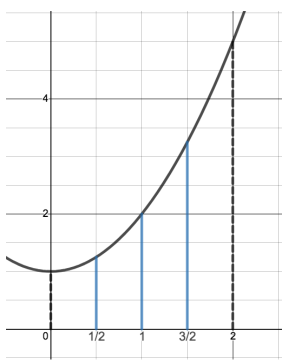
Number of Terms	Riemann Sum
4	5.75
8	5.1875
16	4.921875
32	4.792969
64	4.729492
128	4.697998
256	4.682312
512	4.674484
1024	4.670574
2048	4.66862

Another approach: Use the midpoint as sample point to determine the height of the rectangle.



Number of Terms	Riemann Sum
4	4.625
8	4.65625
16	4.664063
32	4.666016
64	4.666504
128	4.666626
256	4.666656
512	4.666664
1024	4.666666
2048	4.666667

What if the sample point is randomly chosen?



Table

Number of Terms	Riemann Sum
4	4.625
8	4.65625
16	4.664063
32	4.666016
64	4.666504
128	4.666626
256	4.666656
512	4.666664
1024	4.666666
2048	4.666667

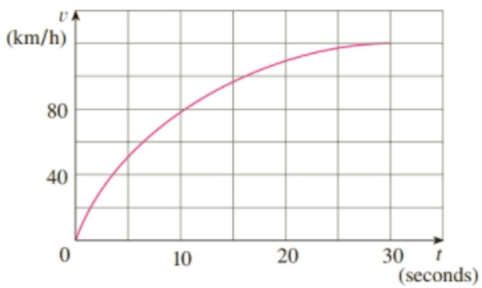
In the next section, we will revisit this problem and use limits to find the area exactly.

The Distance Problem

How can be find the distance traveled by an object if we know its velocity?

But what if the velocity is not constant?

18. The velocity graph of a car accelerating from rest to a speed of 120 km/h over a period of 30 seconds is shown. Estimate the distance traveled during this period.

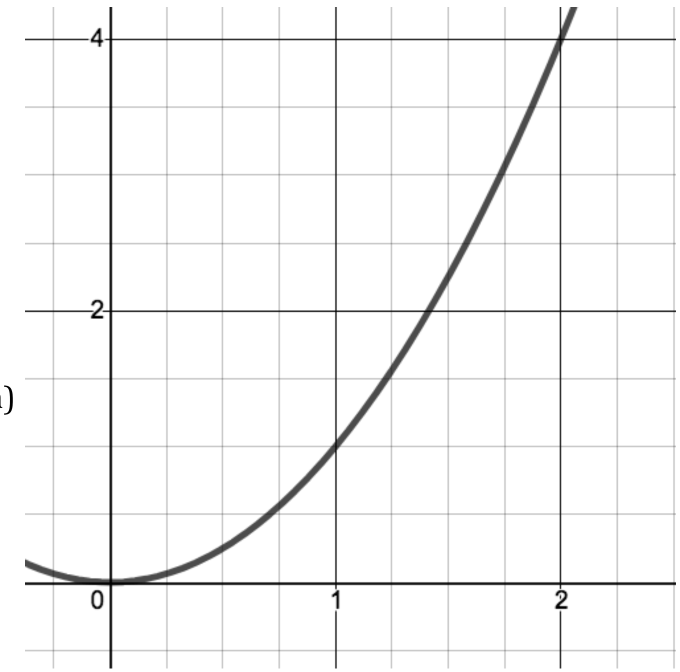


4.2 The Definite Integral

Returning to the example from the last section: Find the area below $f(x) = x^2 + 1$, over [0, 2].

We will use Calculus and Limits to find the area exactly..

<u>Partition</u> the interval [0, 2] into __________ subintervals of equal width____________.



Which divides [0,2] into

$$0 = x_0 < x_1 < x_2 < \cdots \qquad \cdots < x_n = 2$$

where

$$x_0 = \_\_\_\_\_\_; \quad x_1 \_\_\_\_\_\_; \quad x_2 \_\_\_\_\_\_; \quad \cdots x_i \_\_\_\_\_\_; \cdots x_n \_\_\_\_\_\_;$$

Then Rn= (could have used Ln, Mn, random)

So AREA = $\lim\limits_{n\to\infty} R_n = \lim\limits_{n\to\infty}$

How can we write this more compactly? How can we compute it? (See next page)

Summation (Sigma) Notation – Appendix E

1 Definition If $a_m, a_{m+1}, \ldots, a_n$ are real numbers and m and n are integers such that $m \leqslant n$, then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

Sigma notation provides a way to express a long sum.

Example: $\displaystyle\sum_{i=3}^{5} i^2$ $\qquad \displaystyle\sum_{j=0}^{3} 2^j$ $\qquad \displaystyle\sum_{k=1}^{n} x^k$

So we can write $R_n = f\left(\frac{2}{n}\right)\Delta x + f\left(\frac{4}{n}\right)\Delta x + f\left(\frac{6}{n}\right)\Delta x + \cdots + f\left(\frac{2n}{n}\right)\Delta x = \displaystyle\sum_{i=1}^{n}$

Here are some properties that help us manipulate sums written in sigma notation.

5 $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

6 $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

7 $$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

The remaining formulas are simple rules for working with sigma notation:

8 $$\sum_{i=1}^{n} c = nc$$

9 $$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

10 $$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

11 $$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

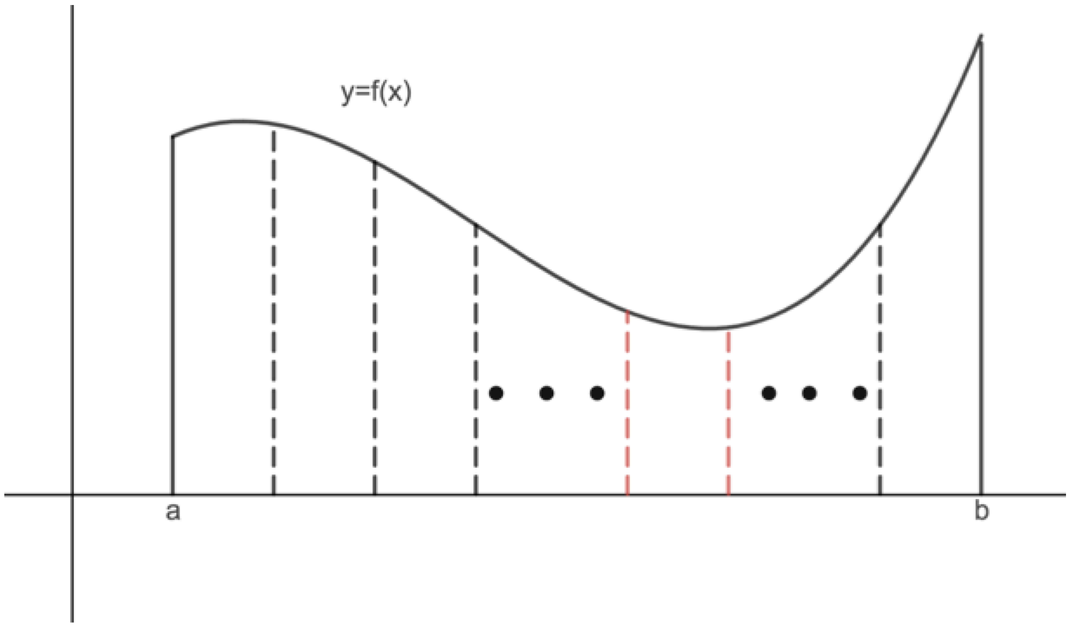
So returning to previous problem…..(see previous page)

Generalizing this process to find the area under any f(x) ≥0, continuous on I=[a, b].

Partition [a,b] into n subintervals of equal width _____________________, which partitions the interval as follows:

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{i-1} < x_i \cdots < x_n = b$$

Where $x_i =$ ____



We form n rectangles, by taking an arbitrary sample point in each subinterval, $x_i^* \in \left[x_i, x_{i-1} \right]$ and using the functional value at that

sample point as the _____________________________ of the rectangle. The area of the i^th rectangle, then, is $\Delta A_i =$ _______________

The area enclosed in the n rectangles is given by: _______________________________________

The above sum is called a Riemann Sum. If we take the limit as the number of subintervals goes to infinity for this Riemann Sum, we get the (exact) area.

AREA =

It turns out, this process useful in *many* applications unrelated to area or distance, so we define the process and notation.

2 Definition of a Definite Integral If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\ (= a), x_1, x_2, \ldots, x_n\ (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

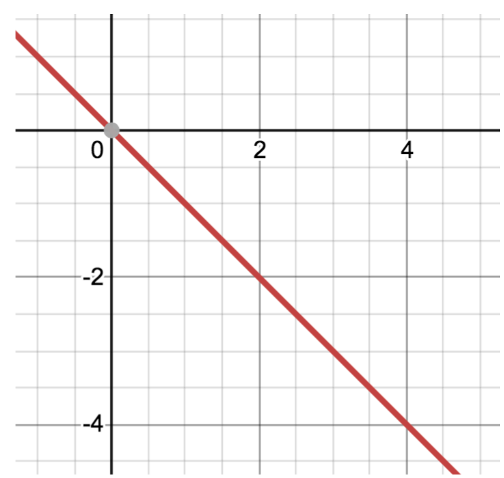
provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

Note, if $f(x) \geq 0$ and continuous, this is the same as the definition of area, but this definition does not make those requirements.

What would this process yield if $f(x) \ngeq 0$ on [a,b]?

Consider applying the process to the function $f(x) = -x$ on [1,5]

Approximate $\int_1^5 f(x)dx$ using L_4



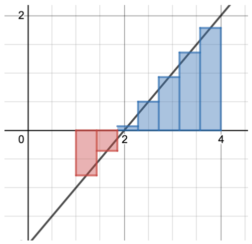
So $\int_1^5 f(x)dx$ is not area. It is RELATED to area. In this case, it is the ____________________ of area.

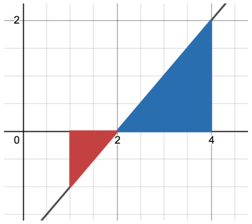
Important: $\int_a^b f(x)dx$ ONLY represents ______________________if _________________

Area is never ____________________, Integrals ARE sometimes ________________

$\int_a^b f(x)dx$ can be thought of being RELATED to area ________________________________

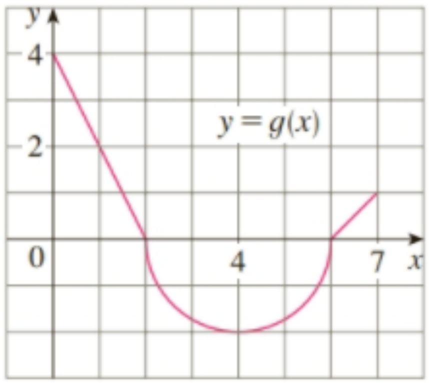
Example Compute using the Riemann sum definition $\int_1^4 (x-2)dx$





34. The graph of g consists of two straight lines and a semicircle. Use it to evaluate each integral.

(a) $\int_0^2 g(x)\,dx$ (b) $\int_2^6 g(x)\,dx$ (c) $\int_0^7 g(x)\,dx$



Suppose you wanted to find the total AREA enclosed between the curve and the x axis in the above problem. What integral expression would g

Methods of Computing an Integral thus far;

1) Approximate using a Riemann Sum
2) Find exactly by finding the limit of a Riemann Sum
3) Use an Area (or "net signed area") interpretation.

$$\int_0^2 \sqrt{4-x^2}\,dx \qquad \int_{-1}^2 (|x|-1)\,dx \qquad \int_{-2}^2 x^3\,dx$$

Properties of Integrals:

6 Integrals of Symmetric Functions Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$.

Properties of the Integral

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant

2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

Example: $\displaystyle\int_0^3 \left(2 + x - 5\sqrt{9 - x^2}\right)dx$

5. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

4.3 The Fundament Theorem of Calculus

Methods of Computing an Integral thus far;

1) Approximate using a Riemann Sum
2) Find exactly by finding the limit of a Riemann Sum
3) Use an Area (or "net signed area") interpretation.

In this section we seek a better technique.

In 4.2 problem #27 we prove that $\int_a^b x\,dx = \frac{b^2 - a^2}{2}$

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function F such that $F' = f$.

Proof: $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} F'(x_i^*)\Delta x$ where $x_i^* \in [x_{i-1}, x_i]$. Consider the interval $[x_{i-1}, x_i]$. Applying the MVT to F(x) on $[x_{i-1}, x_i]$,

This theorem tells us two important things:

Examples:

$$\int_0^2 \left(x^2+1\right) dx$$

$$\int_0^{\pi/2} \cos x \, dx$$

$$\int_1^8 \sqrt[3]{t} \, dt$$

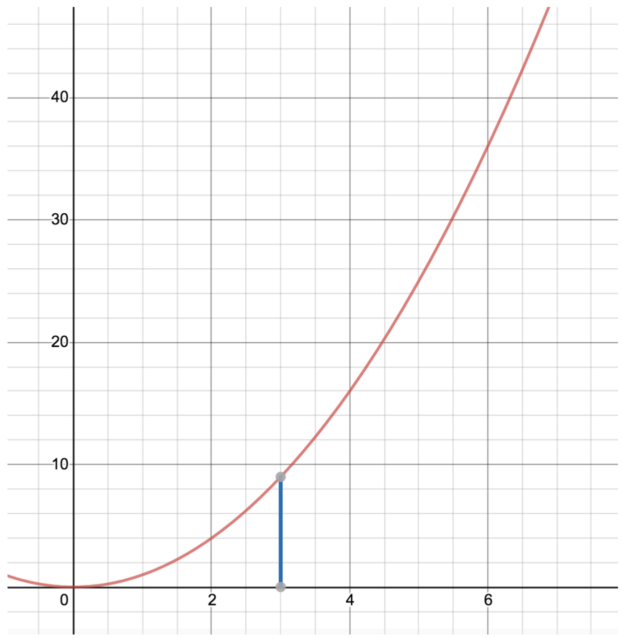
$$\int_1^4 x\sqrt{x} \, dx$$

$$\int_1^9 \frac{x^3+1}{\sqrt{x}} \, dx$$

$$\int_1^2 \frac{1}{x} \, dx$$

Abstract example, motivating the first part of the FTC.

$$\int_3^x t^2 \, dt$$



What does it mean?

How would we compute it?

What would we get if we differentiated now?

So we found that $\dfrac{d}{dx}\displaystyle\int_3^x t^2 \, dt =$

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t) \, dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

That is, $\dfrac{d}{dx}\displaystyle\int_3^x f(t) \, dt = f(x)$

Why is this important?

1) ____

2)____

Example. Find $\dfrac{d}{dx}\displaystyle\int_0^x \tan(t)\,dt; \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$$\frac{d}{dv}\int_0^v \sin(x)\,dx$$

Example: $\dfrac{d}{dx}\displaystyle\int_x^2 \sqrt{t^2+1}\,dt;$

Example involving the Chain Rule.

Back to earlier example

$$\frac{d}{dx}\int_{3}^{x} t^2\,dt$$

Before we learned FTC part 1, we computed it "the long way" by actually integrating and then differentiating.

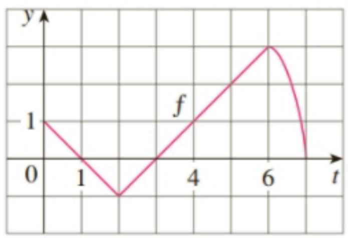
What if we were asked:

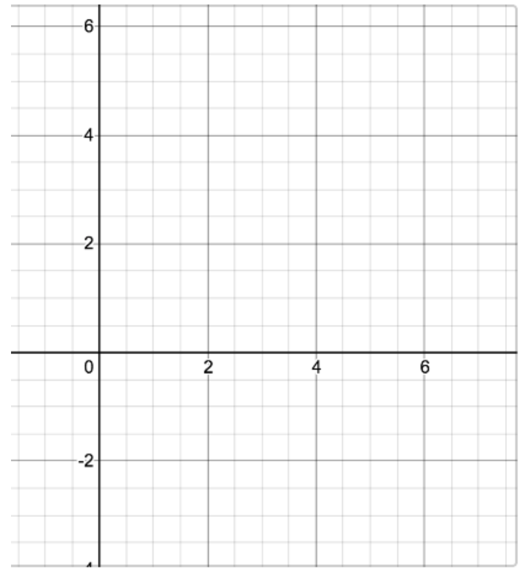
$$\frac{d}{dx}\int_{3}^{\sin x} t^2\,dt$$

In general $\frac{d}{dx}\int_{3}^{u(x)} f(t)\,dt$ will result in a function of u so to differentiate it we will need ________________________________

$$\frac{d}{dx}\int_{3}^{u(x)} f(t)\,dt = f(\underline{\hspace{2cm}})\ \underline{\hspace{2cm}}$$

2. Let $g(x) = \int_0^x f(t)\, dt$, where f is the function whose graph is shown.
   (a) Evaluate $g(x)$ for $x = 0, 1, 2, 3, 4, 5,$ and 6.
   (b) Estimate $g(7)$.
   (c) Where does g have a maximum value? Where does it have a minimum value?
   (d) Sketch a rough graph of g.





The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\, dt$, then $g'(x) = f(x)$.

2. $\int_a^b f(x)\, dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

We noted that Part 1 can be rewritten as

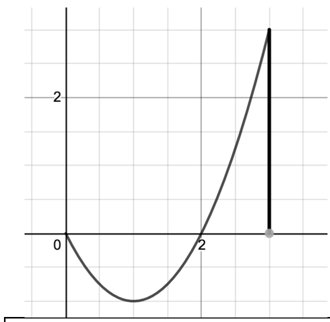
$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$

which says that if f is integrated and then the result is differentiated, we arrive back at the original function f. Since $F'(x) = f(x)$, Part 2 can be rewritten as

$$\int_a^b F'(x)\, dx = F(b) - F(a)$$

4.4 Integrating with Absolute Values – Indefinite Integrals – Net Change Theorem

Lead in example: Find the area enclosed between $f(x) = x^2 - 2x$ and the x axis on [0, 3]



The total area enclosed between f(x) and the x axis over [a, b]: A= $\int_a^b |f(x)|dx$

So for the previous example, $\int_0^3 |x^2 - 2x|dx$. But how would we compute that?

Recall: $|x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$

Compute: $\int_{-1}^{2} |x|\, dx$

Similarly: $|f(x)| = \begin{cases} f(x) & if\ f(x) \geq 0 \\ -f(x) & if\ f(x) < 0 \end{cases}$ so $\int_0^3 |x^2 - 2x|dx$

Net Change

Part 2 of the Fundamental Theorem says that if f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F'(x) is any antiderivative of f, thus F'(x)=f(x). This leads to:

Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

<u>Examples:</u>

- If $V(t)$ is the volume of water in a reservoir at time t, then its derivative $V'(t)$ is the rate at which water flows into the reservoir at time t. So

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$$

is the change in the amount of water in the reservoir between time t_1 and time t_2.

- If the rate of growth of a population is dn/dt, then

$$\int_{t_1}^{t_2} \frac{dn}{dt}\,dt = n(t_2) - n(t_1)$$

is the net change in population during the time period from t_1 to t_2. (The population increases when births happen and decreases when deaths occur. The net change takes into account both births and deaths.)

<u>Displacement vs Distance traveled.</u>

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

2 $$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

is the net change of position, or *displacement*, of the particle during the time period from t_1 to t_2. In Section 4.1 we guessed that this was true for the case where the object moves in the positive direction, but now we have proved that it is always true.

- If we want to calculate the distance the object travels during the time interval, we have to consider the intervals when $v(t) \geqslant 0$ (the particle moves to the right) and also the intervals when $v(t) \leqslant 0$ (the particle moves to the left). In both cases the distance is computed by integrating $|v(t)|$, the speed. Therefore

3 $$\int_{t_1}^{t_2} |v(t)|\,dt = \text{total distance traveled}$$

Example:

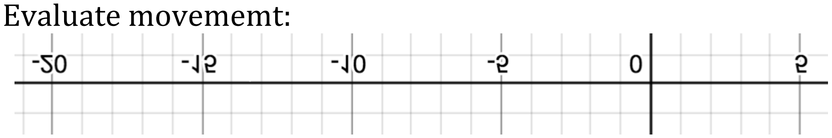
55–56 The velocity function (in meters per second) is given for a particle moving along a line. Find (a) the displacement and (b) the distance traveled by the particle during the given time interval.

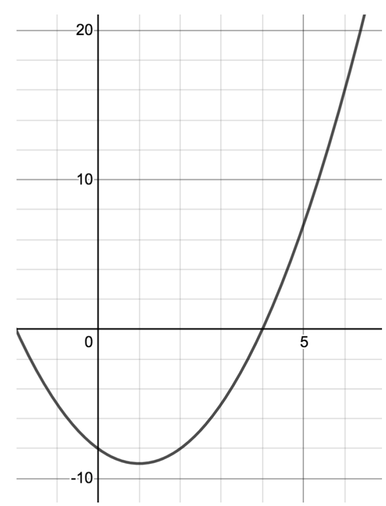
$$v(t) = t^2 - 2t - 8; \quad 1 \le t \le 6$$

Displacement: $\displaystyle\int_{t_1}^{t_2} v(t)dt = \int_1^6 (t^2 - 2t - 8)dt =$

Distance: $\displaystyle\int_{t_1}^{t_2} |v(t)|dt = \int_1^6 |t^2 - 2t - 8|dt =$

Evaluate movememt:





Previously, we considered $\int_a^b f(x)dx$ Note: The result is a ______________________

We now introduce a new notation:

Indefinite Integrals

$\int f(x)dx = F(x)$ where F(x) is the general antiderivative, so will include a +C

Note: The result is a__________________________

Example:

$$\int (x^3 + 2)dx =$$

$$\int \sin x \, dx =$$

$$\int \frac{1}{x^2} \, dx =$$

Book Notation: $\int \frac{1}{x^2} \, dx = \begin{cases} & \text{if } x > 0 \\ & \text{if } x < 0 \end{cases}$

$$\int_1^2 \frac{1}{x^2} dx$$

4.5 Integration by Substitution

So far, the only antiderivatives we can find are:

Today we will learn a new method for helping us find more antiderivatives.

Recall from section 3.9: $g(x) = \sin x$ -----------differentiate---------→ $g'(x) =$ ______________

←----------antidifferentiate -----

or using derivative and integral notation: $\frac{d}{dx}(\sin x) \Longrightarrow$ ________________

________ $\Longleftarrow$ $\int \cos x \, dx$

Suppose f is a composite function $\frac{d}{dx}\left(\tan\left(x^3\right)\right) \Longrightarrow$ ________________

__________ $\Longleftarrow \int$ ______________ dx

$\frac{d}{dx}\left(\sqrt{3x^2+1}\right) \Longrightarrow$ ________________

__________ $\Longleftarrow \int$ ______________ dx

In general, the chain rule tells us $\frac{d}{dx}\left(f(u)\right) \Longrightarrow$ ________________

S0

__________ $\Longleftarrow \int$ ______________ dx

Examples: Choosing u well takes practice. You will begin to "see" two steps ahead whether your choice of u might work.

$$\int 3x^2 \tan\left(x^3\right)dx$$

$$\int x\sqrt{9-x^2}\,dx$$

$$\int 5\cos\left(4x\right)dx$$

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$$

$$\int x\sqrt{x+2}\,dx$$

$$\int \sqrt{x^2+4}\,dx$$

Definite integrals with u-substitution:

$$\int_0^{\pi} \sin(3x)\,dx$$

Changing to U's limits

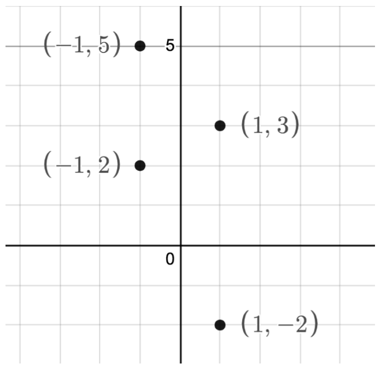
Keeping x's limits: Watch noation!

$$\int_0^1 x^3\left(x^4-2\right)^5 dx$$

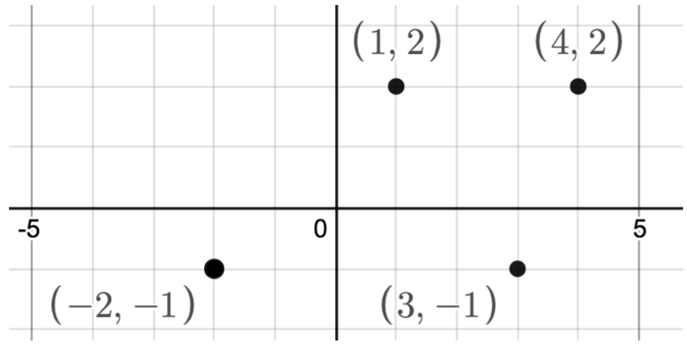
$$\int_{-\pi/2}^{\pi/2} x^3 \sin\left(x^4\right)dx$$

5.1 Area Between Curves (development in book differs)

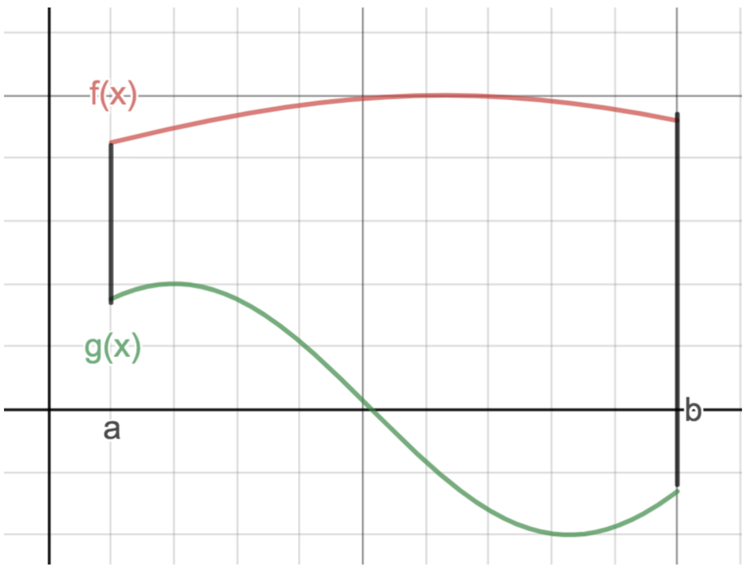
Finding vertical distance:



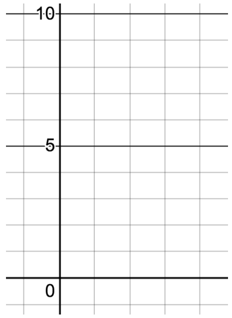
Finding Horizontal distance:



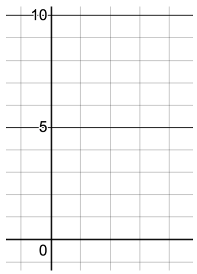
Suppose $f(x)$ and $g(x)$ are continuous on [a, b] with $f(x) \geq g(x)$ for all x in [a, b]. Find the area enclosed between $f(x)$ and $g(x)$ over [a, b].



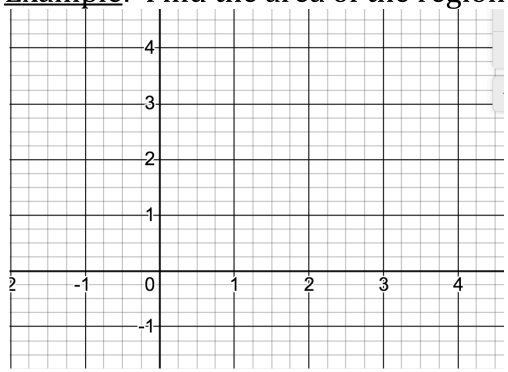
Example: Find the area of the region bounded by $y=x+6$ and $y=x^2$ over $[0,2]$ **(Ans 34/3)**



Example: Find the area of the region bounded by $y=x$ and $y=x^2$. **(Ans 1/6)**

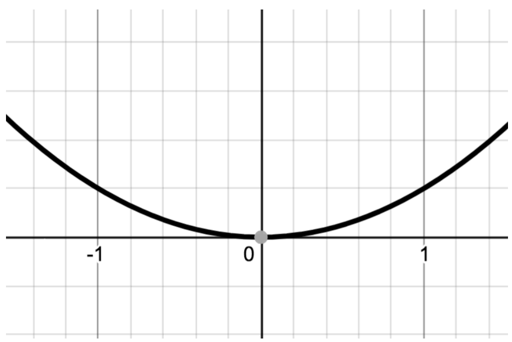


Example: Find the area of the region bounded by $x+y=4$, $y=2$ and $y=x^2+2$ **(Ans 5/6)**

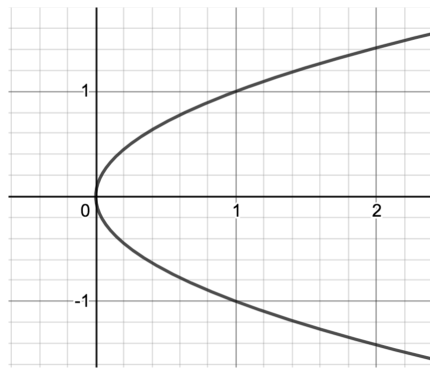


Consider looking at functions from a different point of view:

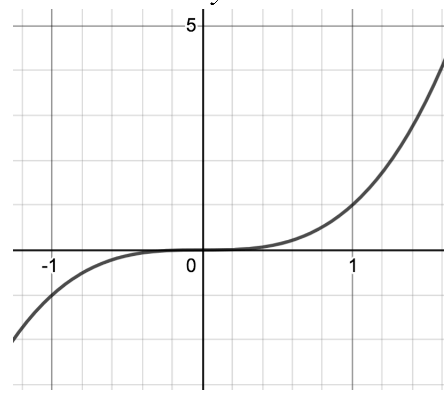
$y = x^2$



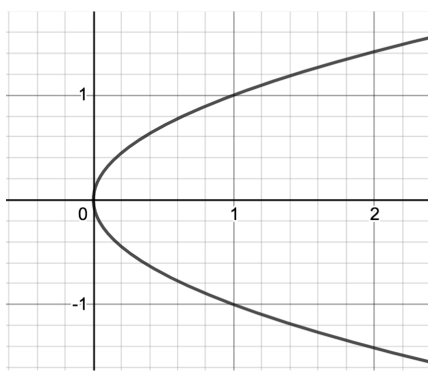
$y^2 = x$



$y = x^3$



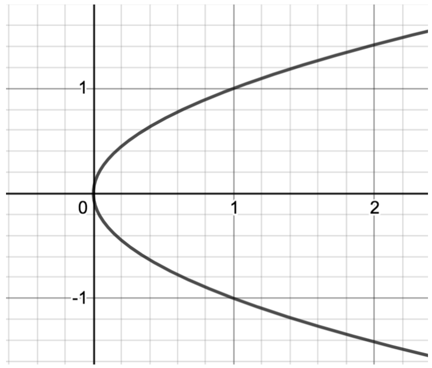
Some curves make more sense to view with y being the independent variable and x being a function of y.



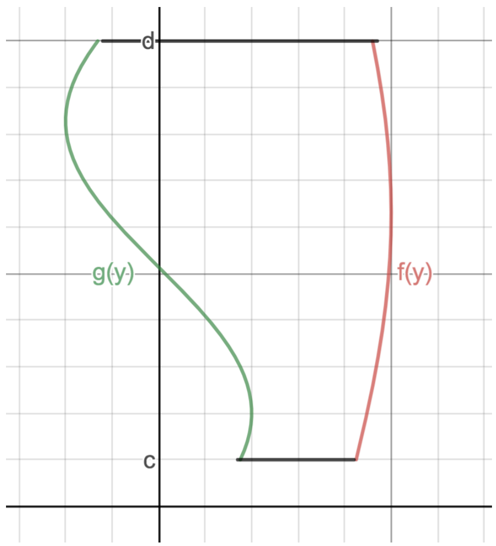
$g(y) = y^2$

x	y

Example: Find the area of the region bounded by $y^2 = x$ and the y axis for $-1 \le y \le 1$ **(Ans 2/3)**



In general suppose $x = f(y)$ and $x = g(y)$ are continuous on $c \le y \le d$ with $f(y) \ge g(y)$ for all y in [c, d]. Find the area enclosed between $f(y)$ and $g(y)$ over [c, d].



Redo example, this time with respect to y: Find the area of the region bounded by $x+y=4$, $y=2$ and $y=x^2+2$